Discrete Mathematics 36 (1981) 171–189
North-Holland Publishing Company

t-DESIGNS FROM THE LARGE MATHIEU GROUPS

Earl S. KRAMER, Spyros S. MAGLIVERAS, and Dale M. MESNER
Department of Mathematics and Statistics, University of Nebraska-Lincoln, Lincoln, NE 68588, USA.

Received 13 March 1979

A t-design $(X, \mathscr{B})$ is a v-set X together with a family $\mathscr{B}$ of k-subsets from X, called blocks, such that each subset of X of size t is contained in exactly λ members of $\mathscr{B}$. A t-design with the above parameters is also called a t-(v, k, λ) design. Here, we allow repeated k-sets in $\mathscr{B}$, i.e. $\mathscr{B}$ is a multiset. We describe the action of the Mathieu groups M_n, $n = 24, 23, 22$, on the power sets of the respective X (Chang Choi and John H. Conway have done this for M_{24}), and then determine all of the quadruples of parameters t, n, k, λ with $2 \leq t < k \leq \frac{1}{2}n$ for which there is a t-(n, k, λ) design with M_n as automorphism group. Among the many new t-designs found there is, for example, an 11-(24, 12, 6) design which is the union of three orbits of 12-sets under M_{24}, two of which are repeated six times.

1. Introduction

A t-design, or t-(v, k, λ) design, is a pair $(X, \mathscr{B})$ where $\mathscr{B}$ is a system of k-sets (called blocks) from a v-set X such that each t-set from X is in exactly λ blocks of $\mathscr{B}$. We shall allow *repeated* blocks. A t-design will be called simple if no blocks are repeated and a t-design will be called trivial iff there is a constant c such that each k-set of X occurs c times in $\mathscr{B}$. Then $\lambda = c\binom{v-t}{k-t}$. An easy necessary condition for the existence of a t-(v, k, λ) design is that

$$\lambda\binom{v-i}{t-i} \equiv 0 \left(\operatorname{mod}\binom{k-i}{t-i}\right) \quad \text{for } i = 0, 1, \ldots, t. \tag{1}$$

Wilson [13] has established: Given integers $v \geq k \geq t \geq 0$, there exists a constant $N = N(t, k, v)$ such that a t-(v, k, λ) exists for all $\lambda \geq N$ satisfying the congruence (1). This is a remarkable result although to-date no simple t-design with $t \geq 6$ has been found. If $t = 4$ or 5 infinite families of t-designs are known (see [1], [7]) and such designs seem to exist in profusion for fairly small v (see [2], [8]). The t-designs with $\lambda = 1$ are called Steiner systems and are often denoted by $S(t, k, v)$. Only a few Steiner systems with $t = 4$ or 5 are known (see [5]). In [8] Kramer has constructed (using repeated blocks) a 6-(17, 7, 2) and a 7-(18, 9, 10), and Denniston (personal communication) has constructed a 6-(28, 7, 6) with repeated blocks. In this paper we construct nonsimple, and nontrivial t-designs for t as large as 11. We accomplish this by using the well-known but fascinating large Mathieu groups. The major effort is to first determine the action of M_n, $n = 24, 23, 22$ on all k-sets of an n-set. Certain matrices are then obtained and in some cases (in particular the 11-(24, 12, 6)) the existence of a t-design becomes transparent.

2. Preliminaries

A well-known property of a t-design $(X, \mathcal{B})$ is that if $x \in X$ and $\mathcal{B}_x = \{B \setminus \{x\} \mid x \in B \in \mathcal{B}\} = D_x(\mathcal{B})$, then $(X \setminus \{x\}, \mathcal{B}_x)$ is a $(t-1)$-$(v-1, k-1, \lambda)$ design called the derived design with respect to x. If $R_x(\mathcal{B}) = \{B \mid x \notin B \in \mathcal{B}\}$, then $R_x(\mathcal{B})$ is the family of blocks for what is called the residual $(t-1)$-design on $X \setminus \{x\}$. If $\mathcal{B}^c = \{X \setminus B \mid B \in \mathcal{B}\}$ then $(X, \mathcal{B}^c)$ is a t-$(v, v-k, \lambda^c)$ where

$$\lambda^c = \binom{k}{t}^{-1} \binom{v-k}{t} \lambda.$$

Hence we can assume $k \leqslant \frac{1}{2}v$.

We let $\mathcal{P}(X)$ be the collection of all subsets of a set X and $\mathcal{P}_k(X)$ the set of all k-subsets of X. If G is a group acting on X, and $x \in X$, $g \in G$, then x^g is the image of x under g; if $Y \subseteq X$, then $Y^g = \{y^g \colon y \in Y\}$ is the image under g of the set Y. Let $\Omega = \Omega_{24} = \{\infty, 0, 1, \ldots, 22\}$, $\Omega_{23} = \Omega \setminus \{\infty\}$, $\Omega_{22} = \Omega_{23} \setminus \{0\}$. Then M_{24} will be represented as a quintuply transitive group on Ω, as in Todd [12]. M_{23} is the subgroup of M_{24} fixing the point ∞, and is quadruply transitive on Ω_{23}. M_{22} is the subgroup of M_{23} fixing the point 0, and is triply transitive on Ω_{22}. $M_{21} \cong \mathrm{PSL}_3(4)$ is the subgroup of M_{22} fixing the point 1, and is doubly transitive on $\Omega_{21} = \Omega_{22} \setminus \{1\}$.

In determining the orbits of a group G on $\mathcal{P}_k(X)$ it is very useful at the outset to compute the number of G-orbits on $\mathcal{P}_k(X)$ for each k. This is possible by an application of a Lemma of Livingstone and Wagner [10], and a character theoretic formulation of Burnside's Lemma (see [4] for more details). Table 1 gives the results of this computation.

Table 1. Number of orbits of k-sets of $G = M_n$ acting on n elements for $n = 24, 23, 22, 21$ and $k \leqslant 12$.

G \ k	1	2	3	4	5	6	7	8	9	10	11	12
M_{24}	1	1	1	1	1	2	2	3	3	3	3	5
M_{23}	1	1	1	1	2	3	4	5	5	5	7	7
M_{22}	1	1	1	2	4	6	8	10	10	13	16	13
M_{21}	1	1	2	5	9	14	22	29	35	45	45	35

We should note that because M_{24} is quintuply transitive, then any orbit of k-sets, $k \geqslant 5$, will be a 5-design. Similarly any orbit of M_{23} is a 4-design, $k \geqslant 4$, and any orbit of M_{22} is a 3-design, $k \geqslant 3$.

3. The action of the large Mathieu groups on an appropriate $\mathcal{P}(X)$

In this section we present the action of M_{24} on $\mathcal{P}(\Omega)$, of M_{23} on $\mathcal{P}(\Omega_{23})$, and of M_{22} on $\mathcal{P}(\Omega_{22})$. Choi [3] and Conway [6] have diagrammed the action of M_{24} on $\mathcal{P}(\Omega)$ and our diagram for the M_{24} action is isomorphic to theirs. Further, we adopt notation so that some strong relationships between the actions of the three groups M_{24}, M_{23}, M_{22} become clear from the diagrams presented.

Conway characterized most of the orbits of $\mathcal{P}(\Omega)$ under M_{24} by their relationships to the orbit of 759 special octads (forming the Steiner system $S(5, 8, 24)$) and to the orbit of 2576 dodecads (called umbral dodecads). We do follow Conway's lead in characterizing orbits by comparing an orbit with one or more selected orbits, but we have not been able to sustain Conway's degree of elegance in notation and terminology. More precisely, we will characterize orbits by means of "frequency vectors". For any set $X \subseteq \Omega_n$ and any orbit $O = {}_nO_i^k$ (where ${}_nO_i^k$ will be the ith orbit of k-subsets of Ω_n under M_n, $n = 24, 23, 22$) the intersection frequency $e_j = e_j(X; O) = |\{Y \in O: |X \cap Y| = j\}|$ of X with respect to O is the number of elements of O which intersect X precisely in j-sets. Then $E = E(X; O) = (e_0, e_1, \ldots, e_k)$ is the frequency vector of X with respect to O. Since $|X \cap Y| = |X^g \cap Y^g|$ for any $g \in M_n$, $E(X; O)$ is constant for all sets X in any orbit O' of M_n and we denote it by $E(O'; O)$.

Now if $O = {}_nO_i^k$ is a 1-(n, k, λ_1) design, which is the case here since M_n is transitive on Ω_n, then $\sum je_j = |X| \lambda_1$ for $E(X; O) = (e_0, \ldots, e_k)$ so $E(X_1; O) \neq E(X_2; O)$ if $|X_1| \neq |X_2|$. Let $O = {}_nO_{i_1}^{k_1}$ be fixed. It is very useful to know the number, say p_k, of orbits of $\mathcal{P}_k(\Omega_n)$ under M_n. For if one finds p_k different sets $X_i^k \in \mathcal{P}_k(\Omega_n)$, where $\mathcal{E}^k(O) = \{E(X_i^k; O): i = 1, \ldots, p_k\}$ and there are p_k distinct frequency vectors in $\mathcal{E}^k(O)$, then $\mathcal{R}^k = \{(X_i^k: i = 1, \ldots, p_k\}$ will be a set of representatives of the orbits of $\mathcal{P}_k(\Omega_n)$ under M_n and $\mathcal{E}^k(O)$ will characterize the orbits ${}_nO_i^k$, i.e. $Y \in {}_nO_i^k$ iff $E(Y; O) = E(X_i^k; O)$. If $O = {}_nO_{i_1}^{k_1}$ has the added virtue that for each k there are p_k sets $X_i^k \in \mathcal{P}_k(\Omega_n)$ for which $|\mathcal{E}^k(O)| = p_k$, then $\{E: E \in \mathcal{E}^k(O), k = 1, \ldots, n\}$ will completely characterize the orbits of $\mathcal{P}(\Omega_n)$ under M_n. For example, if $n = 24$ then choosing $O = {}_{24}O_1^8$ (which is the orbit of 759 special octads) will produce a set of frequency vectors that are in one-to-one correspondence with the orbits of $\mathcal{P}(\Omega)$ under M_{24}. These frequency vectors are given in the Diagram 1 for the action of M_{24} on Ω. The frequency vectors that we used to characterize orbits under M_{23} were $E({}_{23}O_i^k; {}_{23}O_1^7)$ where ${}_{23}O_1^7 = D_\infty({}_{24}O_1^8)$ is the derived design $S(4, 7, 23)$, of $S(5, 8, 24)$, consisting of 253 special heptads. For M_{22}, frequency vectors with respect to the orbit ${}_{22}O_1^6 = D_0({}_{23}O_1^7)$, which is an $S(3, 6, 22)$, did not distinguish between all orbits. For example, $E({}_{22}O_2^6; {}_{22}O_1^6) = E({}_{22}O_3^6; {}_{22}O_1^6) = (6, 36, 15, 20, 0, 0, 0)$. Using ${}_{22}O_2^7 = R_0({}_{23}O_1^7)$ for calculating frequency vectors did provide sufficient discrimination between orbits. Properties of frequency vectors including how some frequency vectors for orbits under M_{23} and M_{22} relate back to frequency vectors for orbits under M_{24} will be discussed later in this section.

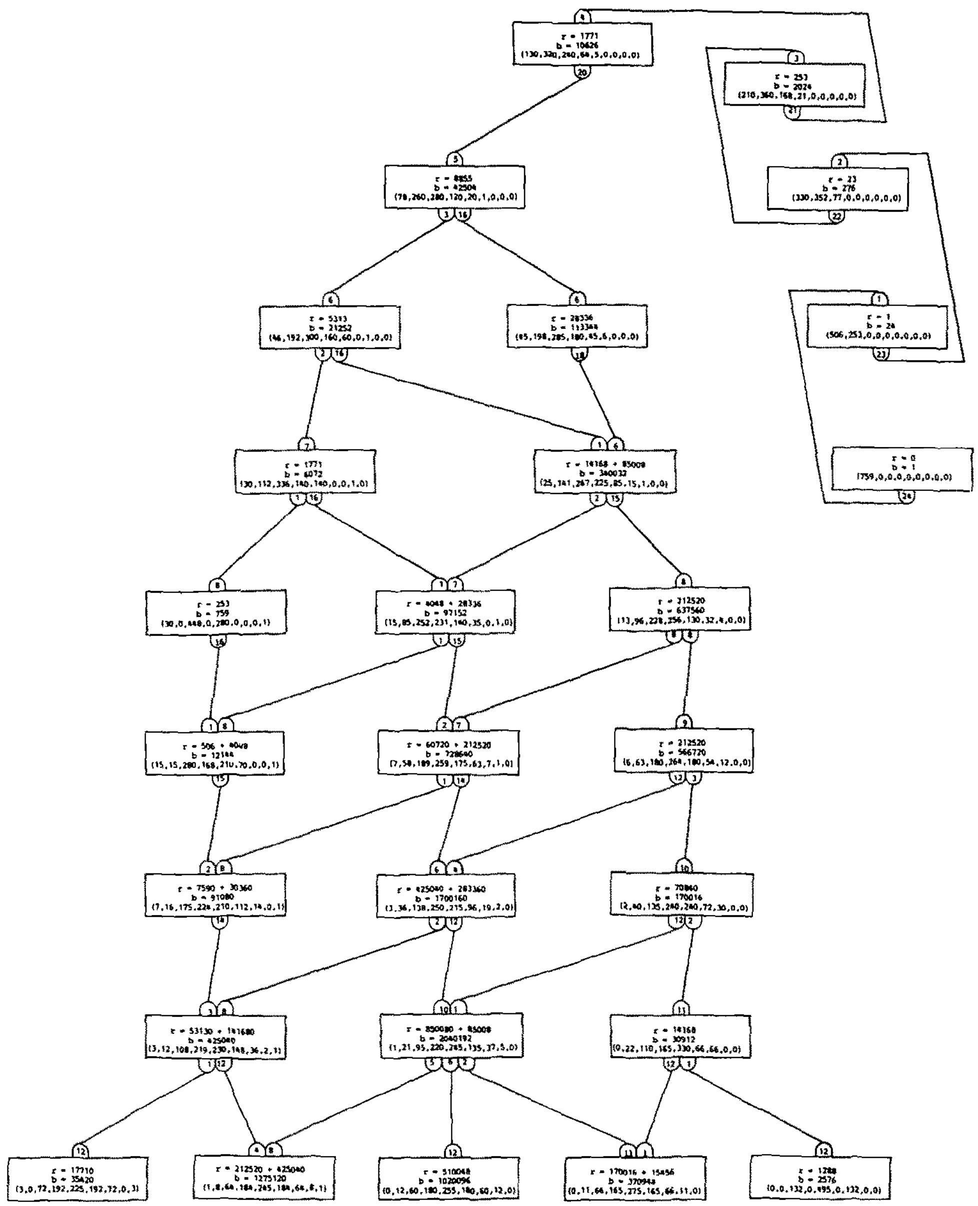

Diagram 1. The action of M_{24} on $P(\Omega)$.

In the diagram for M_n the cells represent orbits, with orbits ${}_nO_i^k$ for fixed k aligned horizontally. For M_{24} a cell for ${}_{24}O_i^k$ gives the orbit length as b (the number of blocks in a 5-$(24, k, \lambda)$); the number r of blocks in the derived design written as the sum of one or more integers; and the frequency vector $E_{24}(O_i^k; {}_{24}O_1^8)$. For M_{23} a cell gives the two numbers b and r, and for M_{22} each cell gives the value of b.

Consider a line that joins an ${}_nO_i^u$ to an ${}_nO_j^{u+1}$. At the end of the line adjacent to

${}_nO_i^u$ there is a number a_{ij} and at the other end there is a number b_{ij}. Then for each $X \in {}_nO_i^u$,

$$a_{ij} = a_{ij}(u, u+1) = |\{Y \in {}_nO_j^{u+1}: X \subset Y\}|$$

and for each $Y \in {}_nO_j^{u+1}$,

$$b_{ij} = b_{ij}(u, u+1) = |\{X \in {}_nO_i^u: X \subset Y\}|.$$

An easy counting argument then implies that

$$a_{ij}(u, u+1)\,|{}_nO_i^u| = b_{ij}(u, u+1)\,|{}_nO_j^{u+1}|.$$

Clearly, each diagram induces a partial ordering of the orbits under M_n where an orbit B is dominated by an orbit A iff for some sets $X \in A$, $Y \in B$ we have that $Y \subset X$.

Now $O^c = \{\Omega_n \setminus X \mid X \in {}_nO_i^k\}$ is an orbit of $(n-k)$-sets under M_n and it is a straightforward exercise to show that the lower half (including the orbit lengths, the a_{ij}'s and b_{ij}'s) of the complete diagram of the action of M_n on $\mathscr{P}(\Omega_n)$ is the mirror image, across the horizontal line where $k = \frac{1}{2}n$, of the upper half of the diagram. Hence our diagrams only include orbits of k-sets for $k \leqslant \frac{1}{2}n$ ($k \leqslant \frac{1}{2}(n+1)$ when $n = 23$).

While the integers a_{ij} and b_{ij} can in many cases be deduced by combinatorial or group-theoretic arguments, it was simple to determine them by computation. If $X \in {}_nO_i^k$ then determining the orbits in which the $(k-1)$-subsets of X lie determines the $b_{ij}(k-1, k)$. Adjoining elements of $\Omega_n \setminus X$ to form $(k+1)$-sets and determining in which orbits they lie determines the $a_{jl}(k, k+1)$. Further, knowing any three of the values in the equation

$$a_{ij}(u, u+1)\,|{}_nO_i^u| = b_{ij}(u, u+1)\,|{}_nO_j^{u+1}|$$

determines the fourth.

Consider, for example, ${}_{24}O_2^8$ where the number of blocks in the derived design $D = D_\infty({}_{24}O_2^8)$ is $r = 4048 + 28336$. Then D is preserved by M_{23} (the stabilizer in M_{24} of the point ∞) and since M_{24} is transitive on Ω the derived designs with respect to any point of Ω are isomorphic. It follows by an easy counting argument that $(b_{12}(7, 8)/k)r = 4048$ of the 7-sets of D are in the residual design $(R_\infty({}_{24}O_1^7)$. In general, if the group is transitive on the n points, then $(b_{ij}(u, u+1)/(u+1))r$ of the u-sets in $D_x({}_nO_j^{u+1})$ are in $R_x({}_nO_i^u)$. Since M_{23} preserves D and $R_\infty({}_{24}O_1^7)$, the 4048 7-sets of $R_\infty({}_{24}O_1^7) \cap D$ are a union of M_{23}-orbits. In fact these 4048 sets are precisely ${}_{23}O_2^7$ and the 28336 remaining sets in D form ${}_{23}O_3^7$. Let $A = {}_{23}O_i^k$, $B = {}_{23}O_j^k$. Then we index the orbits under M_{23} so that if $A \subseteq D({}_{24}O_{i_1}^{k+1})$, $B \subseteq D({}_{24}O_{i_2}^{k+1})$ and $i_1 < i_2$, then $i < j$. If $A \cup B \subseteq D({}_{24}O_{i_1}^{k+1})$ and $A \subseteq R({}_{24}O_{i_3}^k)$, $B \subseteq R({}_{24}O_{i_4}^k)$ with $i_3 < i_4$, then $i < j$. We index similarly for orbits ${}_{22}O_i^k$ under M_{22}, $k \leqslant 11$. (Indexing of orbits ${}_{22}O_i^k$, $k \geqslant 12$, which are not used in this paper, would be modified where necessary to preserve symmetry of Diagram 3 about a

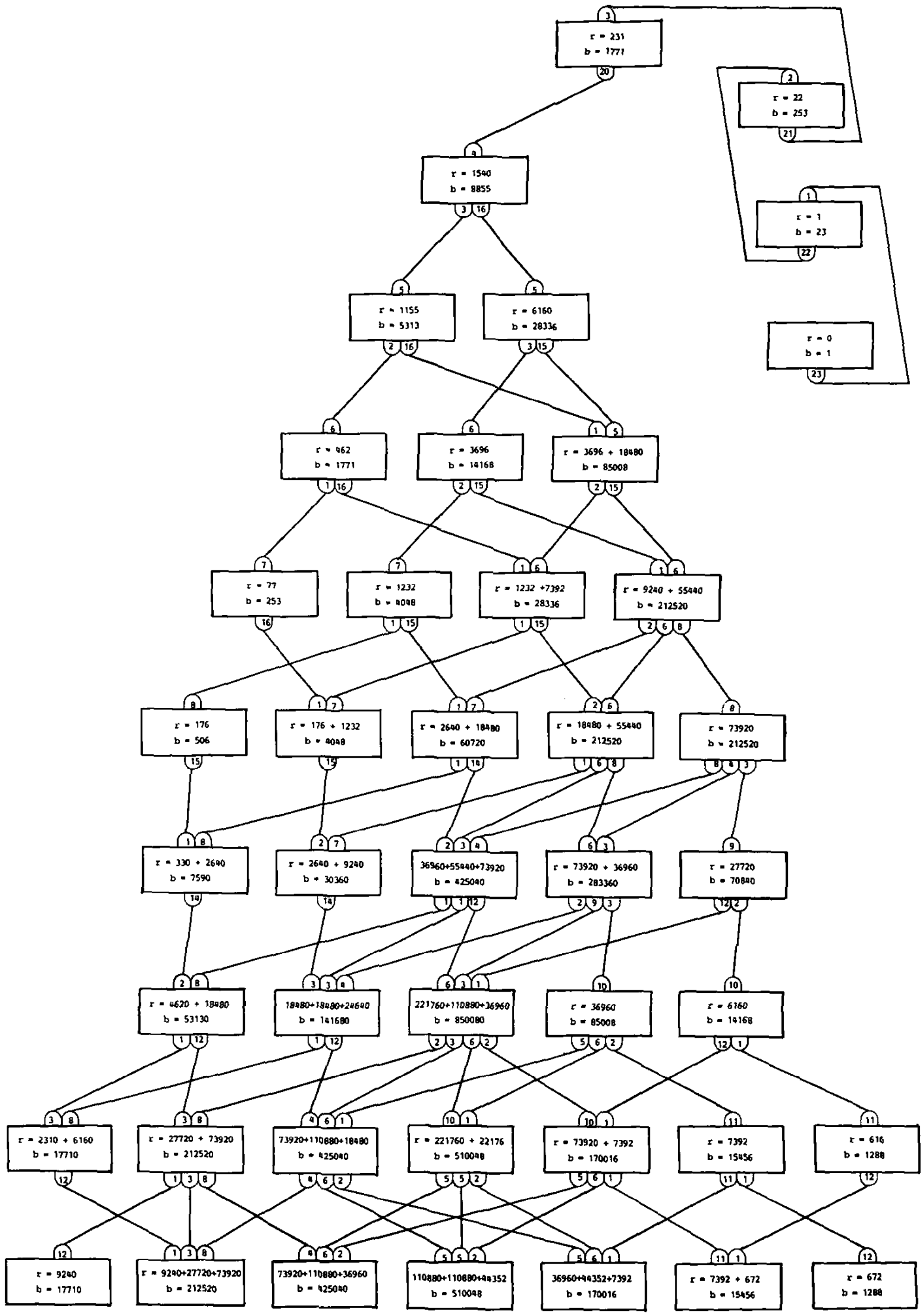

Diagram 2. The action of M_{23} on $P(\Omega_{23})$.

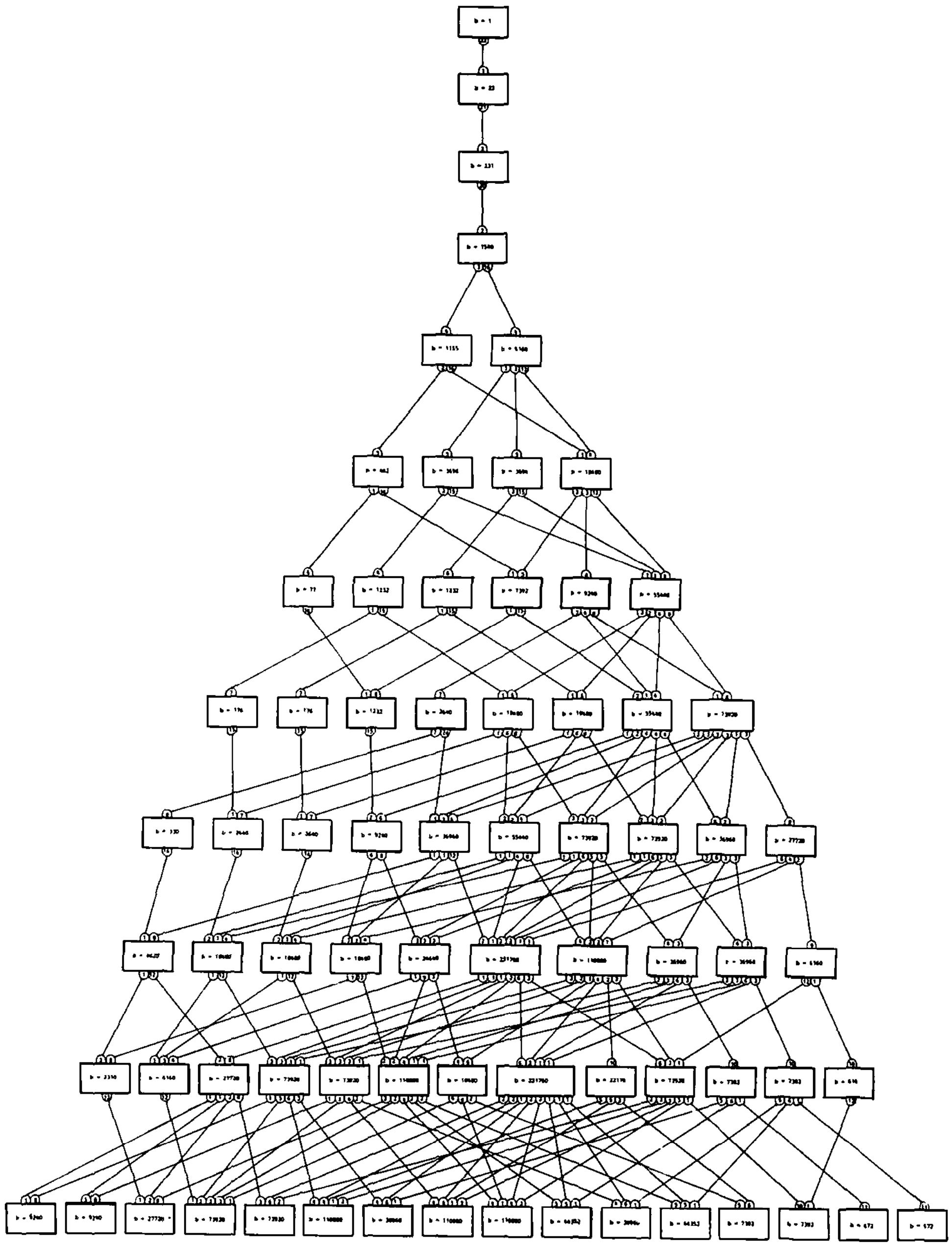

Diagram 3. The action of M_{22} on $P(\Omega_{22})$.

horizontal line at $k=11$.) So, for example,

$$D_\infty({}_{24}O_2^{10}) = {}_{23}O_3^9 \cup {}_{23}O_4^9 \quad \text{and} \quad D_0({}_{23}O_3^9) = {}_{22}O_5^8 \cup {}_{22}O_6^8 \cup {}_{22}O_7^8.$$

For orbits under M_{24} and M_{23} the $b_{ij}r/(u+1)$ sets of size u do correspond to single orbits in the stabilizer of the given group. This pattern does not continue in going from the diagram for M_{22} to that of M_{21}, since there are $6 \cdot 100800/10$ blocks in $D_1({}_{22}O_8^{10})$ which are contained in ${}_{22}O_6^9$ and these 60480 blocks break up into *three* orbits under M_{21}.

Returning to a discussion of frequency vectors, note that by counting in two ways those pairs of sets $X \in O'$, $Y \in O$ such that $|X \cap Y| = i$, then $|O|\,E(O';O) = |O'|\,E(O;O')$ where the vectors may be made conformable by adjoining zero components as needed. Also if $\bar{O}' = \{\Omega \setminus Y \colon Y \in O'\}$ then

$$e_i(\bar{O}'; O_j^m) = e_{m-i}(O'; O_j^m), \quad 0 \leqslant i \leqslant m.$$

Suppose $X \in R_\infty({}_{24}O_i^k) \cap D_\infty({}_{24}O_j^{k+1})$. Then X is a representative of ${}_{24}O_i^k$ and of ${}_{23}O_l^k$ for some l, and $X \cup \{\infty\}$ is a representative of ${}_{24}O_j^{k+1}$. Now ${}_{24}O_1^8 = D \cup {}_{23}O_1^8$ where $D = \{W \cup \{\infty\} \colon W \in {}_{23}O_1^7\}$. Note that ${}_{23}O_1^7 = D_\infty({}_{24}O_1^8) = S(4,7,23)$ is the derived design of ${}_{24}O_1^8 = S(5,8,24)$ and ${}_{23}O_1^8 = R_\infty({}_{24}O_1^8)$ is the residual design of ${}_{24}O_1^8$. Then the blocks Y of ${}_{23}O_1^8$ which have intersection of size β with X also appear as blocks of ${}_{24}O_1^8$, where they still have intersection β with X and have intersection β with $X \cup \{\infty\}$. The blocks Z of ${}_{23}O_1^7$ which have intersection β with X, are in obvious one-to-one correspondence with the blocks of ${}_{24}O_1^8$ which contain ∞ and which have intersection β with X and intersection $\beta+1$ with $X \cup \{\infty\}$. It follows that

$$e_\beta(X; {}_{24}O_1^8) = e_\beta(X; {}_{23}O_1^8) + e_\beta(X; {}_{23}O_1^7),$$

and

$$e_\beta(X \cup \{\infty\}; {}_{24}O_1^8) = e_\beta(X; {}_{23}O_1^8) + e_{\beta-1}(X; {}_{23}O_1^7),$$

where $0 \leqslant \beta \leqslant 8$. If $E(X; {}_{24}O_1^8)$ and $E(X \cup \{\infty\}; {}_{24}O_1^8)$ are known, then $E(X; {}_{23}O_1^8)$ and $E(X; {}_{23}O_1^7)$ are determined. We illustrate schematically the solution to the above system in a particular case. Now

$$E({}_{24}O_3^8; {}_{24}O_1^8) = (13, 96, 228, 256, 130, 32, 4, 0, 0),$$
$$E({}_{24}O_3^9; {}_{24}O_1^8) = (6, 63, 180, 264, 180, 54, 12, 0, 0),$$

and consider

$$\begin{array}{ccccccccccccccccccccccccccccccccccc}
 & & 13 & & & & 96 & & & & 228 & & & & 256 & & & & 130 & & & & 32 & & & & 4 & & & & 0 & & & & 0 \\
 & 6 & & 7 & & 56 & & 40 & & 140 & & 88 & & 176 & & 80 & & 100 & & 30 & & 24 & & 8 & & 4 & & 0 & & 0 & & 0 & & 0 & \\
6 & & & & 63 & & & & 180 & & & & 264 & & & & 180 & & & & 54 & & & & 12 & & & & 0 & & & & 0 & &
\end{array}$$

where it becomes apparent (by referring to the equations and our diagram) that $E({}_{23}O_5^8; {}_{23}O_1^8) = (6, 56, 140, 176, 100, 24, 4, 0, 0)$ and $E({}_{23}O_5^8; {}_{23}O_1^7) = (7, 40, 88,$

$80, 30, 8, 0, 0)$. By this procedure it is clear that from the frequency vectors $E(X; {}_{24}O_1^8)$ we can obtain the frequency vectors $E(X; O)$ where $O = {}_{23}O_1^7$ or ${}_{23}O_1^8$, and we can obtain $E(X; O)$ where $O = {}_{22}O_1^6$ or ${}_{22}O_1^7$ or ${}_{22}O_2^7$ or ${}_{22}O_1^8$.

Observe that if our comparison orbit O, in $E(X; O)$, is a t-(n, k, λ) design then it is an i-(n, k, λ_i) design for $0 \leq i \leq t$. Let $a_i = a_i(X; O) = |\{Y \in O \mid Z \subseteq Y$ where $|Z| = i$ and $Z \subseteq X\}|$, $0 \leq i \leq |X|$. Then easily for $0 \leq i \leq t$, $a_i = \binom{|X|}{i}\lambda_i$. Also

$$e_i = \sum_{j \geq i} (-1)^{j-i} \binom{j}{i} a_j \quad \text{and} \quad a_j = \sum_{i \geq j} \binom{i}{j} e_i.$$

So for determining $E(X; O)$ where $O = {}_{24}O_1^8$, we need only know the size of X and either e_6, e_7, e_8 or a_6, a_7, a_8 for this X. The remaining values are determined by using the values of λ_i, $0 \leq i \leq 5$, and the equations in this paragraph.

Finally, in this section we have presented Table 2 which gives orbit representatives for M_n, $n = 24, 23, 22$.

Table 2

$S \cup \{\infty, 0\}$ is representative for M_{24} for $(k+2)$-sets,

$S \cup \{0\}$ is representative for M_{23} for $(k+1)$-sets,

S is representative for M_{22} for k-sets

Orbit Index For M_{24}	M_{23}	M_{22}	k	Elements of S
1	1	1	4	1 2 3 5
2	2	2	4	1 2 3 4
1	1	1	5	1 2 3 5 14
2	2	2	5	1 2 3 4 7
2	3	3	5	1 2 3 4 6
2	3	4	5	1 2 3 5 15
1	1	1	6	1 2 3 5 14 17
2	2	2	6	1 2 3 4 7 10
2	3	3	6	1 2 3 4 6 15
2	3	4	6	1 2 3 4 5 14
3	4	5	6	1 2 3 4 5 8
3	4	6	6	1 2 3 4 5 6
1	1	1	7	1 2 3 4 7 10 12
1	2	2	7	1 2 3 4 6 15 18
1	2	3	7	1 2 3 4 5 14 17
2	3	4	7	1 2 3 4 5 8 11
2	3	5	7	1 2 3 4 5 7 10
2	4	6	7	1 2 3 4 5 6 15
2	4	7	7	1 2 3 4 5 6 14
3	5	8	7	1 2 3 4 5 8 10
1	1	1	8	1 2 3 4 5 8 11 13
1	1	2	8	1 2 3 4 5 7 10 12
1	2	3	8	1 2 3 4 5 6 15 18
1	2	4	8	1 2 3 4 5 6 14 17
2	3	5	8	1 2 3 4 5 6 9 14
2	3	6	8	1 2 3 4 5 8 10 12
2	3	7	8	1 2 3 4 5 6 8 14
2	4	8	8	1 2 3 4 5 6 8 15
2	4	9	8	1 2 3 4 5 6 7 14
3	5	10	8	1 2 3 4 5 8 10 16
1	1	1	9	1 2 3 4 5 6 8 11 13
1	1	2	9	1 2 3 4 5 6 8 14 20
1	2	3	9	1 2 3 4 5 8 10 12 15
1	2	4	9	1 2 3 4 5 6 7 14 17
1	2	5	9	1 2 3 4 5 6 8 14 17
2	3	6	9	1 2 3 4 5 6 7 9 14
2	3	7	9	1 2 3 4 5 6 8 14 16
2	3	8	9	1 2 3 4 5 6 7 8 14
2	4	9	9	1 2 3 4 5 6 7 11 14
3	5	10	9	1 2 3 4 5 8 10 16 18
1	1	1	10	1 2 3 4 5 6 8 10 11 13
1	1	2	10	1 2 3 4 5 6 8 14 17 20
2	2	3	10	1 2 3 4 5 6 7 8 11 13
2	2	4	10	1 2 3 4 5 6 7 9 10 14
2	3	5	10	1 2 3 4 5 6 7 9 11 14
2	3	6	10	1 2 3 4 5 6 7 8 14 17
2	3	7	10	1 2 3 4 5 6 8 14 16 17
3	4	8	10	1 2 3 4 5 6 7 8 9 14
3	4	9	10	1 2 3 4 5 6 8 14 16 18
4	5	10	10	1 2 3 4 5 6 7 9 13 14
4	5	11	10	1 2 3 4 5 6 7 8 10 14
4	6	12	10	1 2 3 4 5 6 7 11 14 15
5	7	13	10	1 2 3 4 5 8 10 16 18 20
1	1	1	11	1 2 3 4 5 6 7 8 11 13 18
1	2	2	11	1 2 3 4 5 6 7 8 11 12 13
1	2	3	11	1 2 3 4 5 6 7 8 10 11 13
1	2	4	11	1 2 3 4 5 6 7 8 14 17 18
2	3	5	11	1 2 3 4 5 6 7 8 9 11 13
2	3	6	11	1 2 3 4 5 6 7 8 9 14 18
2	3	7	11	1 2 3 4 5 6 7 8 9 10 14
2	4	8	11	1 2 3 4 5 6 7 8 9 11 14
2	4	9	11	1 2 3 4 5 6 7 8 9 14 17
2	4	10	11	1 2 3 4 5 6 7 8 10 14 17
2	5	11	11	1 2 3 4 5 6 7 8 9 14 16
2	5	12	11	1 2 3 4 5 6 7 8 12 14 17
2	5	13	11	1 2 3 4 5 6 8 14 16 17 18
3	6	14	11	1 2 3 4 5 6 7 9 13 14 20
3	6	15	11	1 2 3 4 5 6 7 8 10 14 21
3	7	16	11	1 2 3 4 8 10 13 14 16 17 21

4. Finding t-designs

Our procedure for finding t-designs follows that used in [9] with the major exception that we here allow repeated blocks. We describe this method briefly. If G is a permutation group acting on a v-set X, then let O_i^l be the ith orbit of l-subsets of X. If $\Delta \in O_i^t$ is a fixed member of O_i^t, let a_{ij} be the number of members $\Gamma \in O_j^k$ such that $\Delta \subseteq \Gamma$. That the number a_{ij} is independent of the choice of $\Delta \in O_i^t$ follows from the fact that G acts transitively on O_i^t and O_j^k. Note that a_{ij} would simply be the number of times that a t-subset from O_i^t would occur in the k-sets comprising O_j^k. Let the size of $A = (a_{ij})$ be m by n. Then there is a t-(v, k, λ) design $(X, \mathscr{B})$ with G as an automorphism group iff there is a 1 by n vector x of nonnegative integers such that $A(x^{\mathrm{T}})$ is a column vector of λ's. Let $A(G; X; t, k, v) = A = (a_{ij})$ be the above matrix. Then it is straightforward to show that if $t \leqslant t_1 \leqslant k$, then

$$A(G; X; t, k, v) = \binom{k-t}{k-t_1}^{-1} A(G; X; t, t_1, v) A(G; X; t_1, k, v).$$

Consequently it is only necessary to determine $A(G; X; u, u+1, v)$ for $2 \leqslant u < [\frac{1}{2}v]$ to calculate all required $A(G; X; t, k, v)$. For our situation where $G = M_n$, $n = 24, 23, 22$, the matrix $A(M_n; X; u, u+1, n)$ is easily read from the diagrams giving the group action of M_n, $n = 24, 23, 22$, on $\mathscr{P}_k(X)$. For a given diagram and u we have $A(M_n; \Omega_n; u, u+1, n) = (a_{ij}(u, u+1))$. For example,

$$A(M_{24}; \Omega_{24}; 7, 8, 24) = \begin{bmatrix} 1 & 16 & 0 \\ 0 & 2 & 15 \end{bmatrix},$$

$$A(M_{24}; \Omega_{24}; 8, 9, 24) = \begin{bmatrix} 16 & 0 & 0 \\ 1 & 15 & 0 \\ 0 & 8 & 8 \end{bmatrix},$$

and their product divided by $\binom{9-7}{9-8} = 2$ yields

$$\begin{bmatrix} 16 & 120 & 0 \\ 1 & 75 & 60 \end{bmatrix} = A(M_{24}; \Omega_{24}; 7, 9, 24) = A.$$

Tables 3, 4, and 5 present the matrices $A(M_n; \Omega_n; t, k, n)$ with $(n-19) \leqslant t < k \leqslant [\frac{1}{2}n]$ for $n = 24, 23$, and 22, respectively. Now the row sums of A are $136 = \binom{17}{2}$ or in general $\binom{v-t}{k-t} = \bar{\lambda}$ for a given $A(G; X; t, k, v)$. This $\bar{\lambda}$ would be the λ for the trivial design $(X, \mathscr{B})$ where $\mathscr{B} = \mathscr{P}_k(X)$. Table 6 gives the values of $\bar{\lambda}$ along with the values of $\underline{\lambda}$ where $\underline{\lambda}$ is the smallest positive value of λ satisfying equation (1) for a given choice of t, k, v where $2 \leqslant t < k < \frac{1}{2}v$ and $v = 24, 23, 22$. Clearly any λ for a t-(v, k, λ) must be a multiple of the minimal $\underline{\lambda}$. For the 7-$(24, 9, \lambda)$ the first row of A is divisible by 8 so let $\lambda = 8\alpha$. We get a solution $(4, 0, 1)$ when $\alpha = 8$ and a solution $(0, 4, 3)$ when $\alpha = 60$. The trivial solution $(1, 1, 1)$ has $\alpha = 17$ and by taking appropriate combinations of nontrivial and trivial solutions we can obtain

Table 3. Matrices $A(M_{24}; \Omega_{24}; t, k, 24)$ for $5 \leq t < k \leq 12$.

t \ k	6	7	8	9	10	11	12
5	3 16	3 168	1 128 840	36 2160 1680	540 10080 1008	4620 22176 336	660 23760 19008 6912 48
6		2 16	1 32 120	16 480 320	180 2640 240	1560 6912 96	260 8880 6912 2496 16
		0 18	0 18 135	6 450 360	135 2655 270	1440 7020 108	240 8730 7020 2556 18
7			1 16 0	16 120 0	120 560 0	700 1680 0	140 3360 2016 672 0
			0 2 15	1 75 60	30 590 60	400 1950 30	80 2910 2340 852 6
8				16 0 0	120 0 0	560 0 0	140 1680 0 0 0
				1 15 0	15 105 0	140 420 0	35 945 630 210 0
				0 8 8	4 104 12	88 464 8	22 844 696 256 2
9					15 0 0	105 0 0	35 420 0 0 0
					1 14 0	21 84 0	7 224 168 56 0
					0 12 3	12 90 3	4 198 180 72 1
10						14 0 0	7 84 0 0 0
						2 12 0	1 42 36 12 0
						0 12 2	0 30 36 24 1
11							1 12 0 0 0
							0 5 6 2 0
							0 0 0 12 1

Table 4. Matrices $A(M_{23}; \Omega_{23}; t, k, 23)$ for $4 \leq t < k \leq 11$.

t \ k	5	6	7	8	9	10	11
4	3 16	3 24 144	1 16 112 840	4 32 480 1680 1680	108 432 6048 4032 1008	1260 3360 20160 2016 336	660 7920 15840 19008 6336 576 48
5		2 0 16	1 0 32 120	0 16 80 400 320	20 160 1520 1120 240	360 1200 6240 672 96	260 2800 6080 6912 2272 224 16
		0 3 15	0 3 15 135	1 5 105 345 360	30 105 1605 1050 270	405 1035 6390 630 108	240 2940 5790 7020 2346 210 18
6			1 0 16 0	0 16 0 120 0	0 120 240 320 0	60 640 1440 240 0	140 720 2640 2016 576 96 0
			0 2 0 15	1 0 30 45 60	15 15 390 200 60	150 250 1800 150 30	80 1080 1830 2340 792 60 6
			0 0 2 15	0 1 15 60 60	5 25 350 240 60	105 295 1770 180 30	80 960 1950 2340 780 72 6
7				0 16 0 0 0	0 120 0 0 0	0 560 0 0 0	140 0 1680 0 0 0 0
				1 0 15 0 0	15 0 105 0 0	105 35 420 0 0	35 525 420 630 210 0 0
				0 1 0 15 0	0 15 45 60 0	15 125 360 60 0	35 225 720 630 180 30 0
				0 0 2 6 8	1 3 64 40 12	26 62 424 40 8	22 290 554 696 236 20 2
8					15 0 0 0 0	105 0 0 0 0	35 420 0 0 0 0 0
					0 15 0 0 0	0 105 0 0 0	35 0 420 0 0 0 0
					1 0 14 0 0	14 7 84 0 0	7 112 112 168 56 0 0
					0 1 6 8 0	3 18 72 12 0	7 60 164 168 48 8 0
					0 0 8 4 3	4 8 84 6 3	4 72 126 180 68 4 1
9						14 0 0 0 0	7 84 0 0 0 0 0
						0 14 0 0 0	7 0 84 0 0 0 0
						1 1 12 0 0	1 18 24 36 12 0 0
						0 2 9 3 0	1 9 33 36 9 3 0
						0 0 12 0 2	0 12 18 36 24 0 1
10							1 12 0 0 0 0 0
							1 0 12 0 0 0 0
							0 2 3 6 2 0 0
							0 0 5 6 0 2 0
							0 0 0 0 12 0 1

Table 5. Matrices $A(M_{22}; \Omega_{22}; t, k, 22)$ for $3 \leqslant t < k \leqslant 11$.

t \ k	4		5				6						7								8									
3	3	16	3	24	24	120	1	16	16	96	120	720	4	4	28	60	420	420	1260	1680	12	96	96	336	1344	2016	2688	2688	1344	1008
4			2	0	0	16	1	0	0	32	24	96	0	0	16	16	64	64	336	320	4	16	16	144	384	432	704	704	416	240
			0	3	3	12	0	3	3	12	18	117	1	1	4	12	93	93	252	360	3	27	27	78	348	549	708	708	342	270
5							1	0	0	16	0	0	0	0	16	0	0	0	120	0	0	0	0	120	80	0	160	160	160	0
							0	2	0	0	0	15	1	0	0	0	30	15	30	60	0	15	5	10	60	150	180	140	60	60
							0	0	2	0	0	15	0	1	0	0	15	30	30	60	0	5	15	10	60	150	140	180	60	60
							0	0	0	2	3	12	0	0	1	3	12	12	48	60	1	4	4	21	86	108	156	156	84	60
6													0	0	16	0	0	0	0	0	0	0	0	120	0	0	0	0	0	0
													1	0	0	0	15	0	0	0	0	15	0	0	0	45	60	0	0	0
													0	1	0	0	0	15	0	0	0	0	15	0	0	45	0	60	0	0
													0	0	1	0	0	0	15	0	0	0	0	15	15	0	30	30	30	0
													0	0	0	2	0	0	6	8	1	0	0	3	28	12	24	24	16	12
													0	0	0	0	2	2	4	8	0	1	1	2	12	24	28	28	12	12
7																					0	15	0	0	0	0	0	0	0	0
																					0	0	15	0	0	0	0	0	0	0
																					0	0	0	15	0	0	0	0	0	0
																					1	0	0	0	14	0	0	0	0	0
																					0	1	0	0	0	6	8	0	0	0
																					0	0	1	0	0	6	0	8	0	0
																					0	0	0	1	2	0	4	4	4	0
																					0	0	0	0	2	3	3	3	1	3

t \ k	9									
3	252	1008	1008	1008	1344	12096	6048	2016	2016	336
4	88	272	272	416	512	3840	1728	672	672	96
	78	327	327	300	408	3816	1944	630	630	108
5	20	40	40	280	320	960	240	240	240	0
	15	135	90	60	100	1020	600	180	150	30
	15	90	135	60	100	1020	600	150	180	30
	25	80	80	95	120	1080	510	180	180	30
6	0	0	0	240	320	0	0	0	0	0
	0	105	15	0	20	180	180	60	0	0
	0	15	105	0	20	180	180	0	60	0
	5	10	10	55	60	240	60	60	60	0
	14	12	12	26	24	288	96	40	40	8
	4	22	22	16	24	248	136	40	40	8
7	0	105	0	0	0	0	0	0	0	0
	0	0	105	0	0	0	0	0	0	0
	0	0	0	45	60	0	0	0	0	0
	14	0	0	7	0	84	0	0	0	0
	0	14	3	0	4	36	36	12	0	0
	0	3	14	0	4	36	36	0	12	0
	1	2	2	8	8	48	12	12	12	0
	1	3	3	2	3	51	27	6	6	3
8	14	0	0	0	0	0	0	0	0	0
	0	14	0	0	0	0	0	0	0	0
	0	0	14	0	0	0	0	0	0	0
	0	0	0	6	8	0	0	0	0	0
	1	0	0	1	0	12	0	0	0	0
	0	1	1	0	0	4	8	0	0	0
	0	1	0	0	1	6	3	0	3	0
	0	0	1	0	1	6	3	3	0	0
	0	0	0	2	0	6	0	3	3	0
	0	0	0	0	0	8	4	0	0	2

t \ k	10												
3	180	480	2160	5760	5760	8640	1440	17280	1728	5760	576	576	48
4	84	176	816	1984	1984	3552	544	6336	576	2048	224	224	16
	63	177	792	2148	2148	3114	528	6372	648	2136	210	210	18
5	60	80	240	480	480	1920	240	1920	96	480	96	96	0
	15	65	240	840	720	930	180	2100	240	720	72	60	6
	15	65	240	720	840	930	180	2100	240	720	60	72	6
	24	56	276	684	684	1092	174	2136	204	708	72	72	6
6	60	80	0	0	0	1440	240	0	0	0	0	0	0
	0	35	45	480	180	180	60	540	90	180	30	0	0
	0	35	45	180	480	180	60	540	90	180	0	30	0
	15	20	75	150	150	510	60	600	30	150	30	30	0
	10	12	114	176	176	336	42	648	48	216	20	20	2
	5	17	74	216	216	286	52	628	68	216	20	20	2
7	0	35	0	420	0	0	0	0	0	0	0	0	0
	0	35	0	0	420	0	0	0	0	0	0	0	0
	15	20	0	0	0	360	60	0	0	0	0	0	0
	7	0	84	28	28	84	0	168	0	56	0	0	0
	0	7	12	100	48	48	16	144	24	48	8	0	0
	0	7	12	48	100	48	16	144	24	48	0	8	0
	3	4	20	40	40	112	12	160	8	40	8	8	0
	1	3	21	51	51	63	12	162	18	64	4	4	1
8	7	0	84	0	0	0	0	0	0	0	0	0	0
	0	7	0	84	0	0	0	0	0	0	0	0	0
	0	7	0	0	84	0	0	0	0	0	0	0	0
	3	4	0	0	0	72	12	0	0	0	0	0	0
	1	0	12	6	6	18	0	36	0	12	0	0	0
	0	1	2	16	16	4	4	28	8	12	0	0	0
	0	1	3	15	6	15	3	33	3	9	3	0	0
	0	1	3	6	15	15	3	33	3	9	0	3	0
	1	0	3	6	6	27	0	36	0	6	3	3	0
	0	0	4	8	8	8	2	32	4	24	0	0	1
9	1	0	12	0	0	0	0	0	0	0	0	0	0
	0	1	0	12	0	0	0	0	0	0	0	0	0
	0	1	0	0	12	0	0	0	0	0	0	0	0
	1	0	0	0	0	12	0	0	0	0	0	0	0
	0	1	0	0	0	9	3	0	0	0	0	0	0
	0	0	1	1	1	2	0	6	0	2	0	0	0
	0	0	2	2	2	0	1	4	2	2	0	0	0
	0	0	0	2	0	3	0	6	0	0	2	0	0
	0	0	0	0	2	3	0	6	0	0	0	2	0
	0	0	0	0	0	0	0	0	0	12	0	0	1

t \ k	11															
3	990	990	2970	7920	7920	11880	3960	11880	11880	4752	3960	4752	792	792	72	72
4	400	400	1392	3328	3328	4800	1632	4800	5184	2112	1632	2112	320	320	32	32
	420	420	1224	3336	3336	5040	1674	5040	4968	1980	1674	1980	336	336	30	30
5	120	120	840	1440	1120	1440	560	1440	2400	1056	560	1056	112	80	16	16
	180	160	420	1340	1260	2100	690	1980	1860	732	640	720	140	132	12	10
	160	180	420	1340	1260	1980	640	2100	1860	720	690	732	140	132	10	12
	160	160	504	1276	1316	1920	644	1920	1968	792	644	792	126	130	12	12
6	0	0	720	960	0	0	0	0	1440	576	0	576	96	0	0	0
	105	45	90	600	360	990	330	630	540	216	180	180	60	36	6	0
	45	105	90	600	360	630	180	990	540	180	330	216	60	36	0	6
	45	45	270	480	420	540	210	540	810	360	210	360	36	30	6	6
	58	58	204	408	528	648	220	648	708	288	220	288	36	48	4	4
	58	58	156	456	456	708	232	708	672	264	232	264	48	48	4	4
7	105	0	0	420	0	630	210	0	0	0	0	0	0	0	0	0
	0	105	0	420	0	0	0	630	0	0	210	0	0	0	0	0
	0	0	225	300	0	0	0	0	450	180	0	180	30	0	0	0
	28	28	105	84	280	168	56	168	210	84	56	84	0	14	0	0
	28	15	30	172	120	288	96	210	180	72	60	60	20	12	2	0
	15	28	30	172	120	210	60	288	180	60	96	72	20	12	0	2
	15	15	75	140	140	180	70	180	240	108	70	108	10	10	2	2
	18	18	45	129	155	228	71	228	210	78	71	78	15	19	1	1
8	28	28	84	0	224	0	0	0	0	0	0	0	0	0	0	0
	28	0	0	112	0	168	56	0	0	0	0	0	0	0	0	0
	0	28	0	112	0	0	0	168	0	0	56	0	0	0	0	0
	0	0	60	80	0	0	0	0	120	48	0	48	8	0	0	0
	6	6	24	24	64	48	16	48	60	24	16	24	0	4	0	0
	6	6	4	44	32	76	20	76	44	12	20	12	8	4	0	0
	6	3	12	39	36	66	26	48	57	27	15	21	4	3	1	0
	3	6	12	39	36	48	15	66	57	21	26	27	4	3	0	1
	3	3	24	30	36	42	21	42	66	36	21	36	0	2	1	1
	4	4	8	24	48	68	16	68	60	16	16	16	4	12	0	0
9	6	6	18	0	48	0	0	0	0	0	0	0	0	0	0	0
	6	0	0	24	0	36	12	0	0	0	0	0	0	0	0	0
	0	6	0	24	0	0	0	36	0	0	12	0	0	0	0	0
	0	0	18	12	0	0	0	0	24	12	0	12	0	0	0	0
	0	0	9	21	0	0	0	0	27	9	0	9	3	0	0	0
	1	1	3	5	12	12	4	12	13	5	4	5	0	1	0	0
	1	1	0	8	6	18	4	18	10	2	4	2	3	1	0	0
	1	0	3	6	6	12	10	6	12	12	3	6	0	0	1	0
	0	1	3	6	6	6	3	12	12	6	10	12	0	0	0	1
	0	0	0	0	12	18	0	18	18	0	0	0	0	12	0	0
10	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	0	0	8	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	3	0	6	2	0	0	0	0	0	0	0	0	0
	0	1	0	3	0	0	0	6	0	0	2	0	0	0	0	0
	0	0	2	2	0	0	0	0	4	2	0	2	0	0	0	0
	0	0	0	4	0	0	0	0	6	0	0	0	2	0	0	0
	0	0	0	2	0	2	1	0	0	1	1	1	0	0	0	0
	0	0	0	0	0	5	0	5	0	0	0	0	0	2	0	0
	0	0	0	0	2	3	0	3	3	0	0	0	0	0	1	0
	0	0	0	0	0	0	5	0	0	6	0	0	0	0	0	1
	0	0	0	0	0	0	0	0	0	0	5	6	0	0	1	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	12	0	0

Table 6. For given v, t, k, a cell contains $\underline{\lambda}=(\text{minimal }\lambda)$ as the upper entry and $\bar{\lambda}=\binom{v-t}{k-t}$ as the lower entry.

v	t \ k	3	4	5	6	7	8	9	10	11	12
24	2	2 22	3 231	20 1540	5 7315	42 26334	7 74613	24 170544	45 319770	110 497420	11 646646
	3		3 21	30 210	10 1330	105 5985	21 20349	84 54264	180 116280	45 203490	5 293930
	4			20 20	10 190	20 1140	5 4845	24 15504	60 38760	120 77520	15 125970
	5				1 19	3 171	1 969	6 3876	18 11628	42 27132	6 50388
	6					6 18	3 153	24 816	90 3060	252 8568	42 18564
	7						1 17	4 136	20 680	70 2380	14 6188
	8							8 16	60 120	280 560	70 1820
	9								15 15	105 105	35 455
	10									14 14	7 91
	11										1 13
23	2	3 21	6 210	10 1330	15 5985	21 20349	28 54264	36 116280	45 203490	5 293930	
	3		4 20	10 190	20 1140	5 4845	8 15504	12 38760	120 77520	15 125970	
	4			1 19	3 171	1 969	2 3876	18 11628	42 27132	6 50388	
	5				6 18	3 153	8 816	90 3060	252 8568	42 18564	
	6					1 17	4 136	20 680	70 2380	14 6188	
	7						8 16	60 120	280 560	70 1820	
	8							15 15	105 105	35 455	
	9								14 14	7 91	
	10									1 13	
22	2	2 20	2 190	20 1140	5 4845	2 15504	4 38760	24 77520	15 125970	10 167960	
	3		1 19	3 171	1 969	1 3876	6 11628	42 27132	6 50388	9 75582	
	4			6 18	3 153	4 816	30 3060	252 8568	42 18564	72 31824	
	5				1 17	2 136	20 680	70 2380	14 6188	28 12376	
	6					4 16	60 120	280 560	70 1820	168 4368	
	7						15 15	105 105	35 455	105 1365	
	8							14 14	7 91	28 364	
	9								1 13	6 78	
	10									12 12	

nontrivial 7-(24, 9, 8α)'s for all $\alpha \geqslant 120$. The matrix B for 8-(24, 9, λ) is nonsingular so any solutions will be multiples of the trivial one (1, 1, 1), i.e. we can only get trivial solutions.

Suppose that $B = A(G; X; t, t_1, v)$, $A = A(G; X; t_1, k, v)$, and

$$C = \beta^{-1}BA = \binom{k-t}{t_1-t}^{-1} BA = A(G; X; t, k, v)$$

where B is nonsingular with row sums $\bar{\lambda} = \binom{v-t}{t_1-t}$. If $J = [1\ 1 \cdots 1]^{\mathrm{T}}$, then it easily follows that x is a solution of $Ax^{\mathrm{T}}=\lambda J$ if and only if x is a solution of $Cx^{\mathrm{T}}=\lambda\bar{\lambda}\beta^{-1}J$. This would mean, for the given G and X, that any t_1-(v, k, λ) is a t-$(v, k, \lambda\bar{\lambda}\beta^{-1})$, and conversely. We denote this by t_1-$(v, k, \lambda)\Leftrightarrow t$-$(v, k, \lambda\bar{\lambda}\beta^{-1})$. For example, with the B and A mentioned in the previous paragraph it readily follows that 7-$(24, 9, 8\alpha)\Leftrightarrow$6-$(24, 9, 48\alpha)$.

Consider the matrix $D=A(M_{24}; \Omega; 11, 12, 24)$, namely

$$D=\begin{bmatrix} 1 & 12 & 0 & 0 & 0 \\ 0 & 5 & 6 & 2 & 0 \\ 0 & 0 & 0 & 12 & 1 \end{bmatrix}.$$

It is transparent that there is an 11-(24, 12, 6) with solution (6, 0, 1, 0, 6). Other solutions from D and the other matrices were found almost exclusively by hand (with varying degrees of difficulty) and we simply present them here. If a nontrivial solution for a quadruple t-(n, k, λ) is not obtainable by combinations of those we present together with trivial solutions, then there does not exist a t-(n, k, λ)-design with M_n as automorphism group. When a solution does exist for some λ we have not concerned ourselves with possible isomorphism types.

Designs with M_{24}

Here 11-$(24, 12, \lambda)\Leftrightarrow$10-$(24, 12, 7\lambda)\Leftrightarrow$9-$(24, 12, 35\lambda)\Leftrightarrow$8-$(24, 12, 140\lambda)$. For 11-$(24, 12, \lambda)$ we get $\lambda=6$ with (6, 0, 1, 0, 6), $\lambda=14$ with (14, 0, 2, 1, 2), and $\lambda=17$ with (5, 1, 2, 0, 17). Using these solutions along with the trivial solution (1, 1, 1, 1, 1) for $\lambda=13$ we obtain *nontrivial* solutions for $\lambda=6, 12, 14, 17, 18, 19, 20$ and all $\lambda\geqslant 23$. Now here 7-$(24, 12, \lambda)\Leftrightarrow$6-$(24, 12, 3\lambda)$ and the first row of the matrix for 7-$(24, 12, \lambda)$ implies $\lambda=28\alpha$. We get $\alpha=5$ with (1, 0, 0, 0, 10), $\alpha=39$ with (3, 0, 0, 1, 0), and $\alpha=102$ with (6, 0, 1, 0, 6). These eventually produce all $\alpha\geqslant 137$. Here 5-$(24, 12, 12\alpha)\Leftrightarrow$4-$(24, 12, 30\alpha)\Leftrightarrow$3-$(24, 12, 70\alpha)\Leftrightarrow$2-$(24, 12, 154\alpha)$ where the matrix for 5-$(24, 12, \lambda)$ forces $\lambda=12\alpha$. One gets $\alpha=4$ with (0, 0, 0, 0, 1), $\alpha=55$ with (1, 0, 0, 0, 0) and eventually all $\alpha\geqslant 162$.

When $k=11$ the designs are trivial when $t=8, 9, 10$. When $t=7$ the first row implies $\lambda=140\alpha$. Also 7-$(24, 11, 140\alpha)\Leftrightarrow$6-$(24, 11, 504\alpha)$. We get $\alpha=5$ with (1, 0, 10) and with the trivial design $\alpha=7$ we get all nontrivial designs with $\alpha\geqslant 29$. Now the matrix for $t=5$ implies $\lambda=84\alpha$. Also 5-$(24, 11, 84\alpha)\Leftrightarrow$4-$(24, 11, 240\alpha)\Leftrightarrow$3-$(24, 11, 630\alpha)\Leftrightarrow$2-$(24, 11, 1540\alpha)$. We obtain $\alpha=4$ with (0, 0, 1), $\alpha=55$ with (1, 0, 0) and then all $\alpha\geqslant 162$.

When $k=10$ the designs are trivial when $t=8$ and 9. For $t=7$ the first row implies $\lambda=40\alpha$. Here 7-$(24, 10, 40\alpha)\Leftrightarrow$6-$(24, 10, 180\alpha)$. We get $\alpha=6$ with (2, 0, 3), which, with the trivial solution, then produces all nontrivial solutions with $\alpha\geqslant 86$. When $t=5$ the matrix forces $\lambda=36\alpha$. Also 5-$(24, 10, 36\alpha)\Leftrightarrow$4-$(24, 10, 120\alpha)\Leftrightarrow$3-$(24, 10, 360\alpha)\Leftrightarrow$2-$(24, 10, 990\alpha)$. We get $\alpha=15$ with (1, 0, 0), $\alpha=28$ with (0, 0, 1) and eventually all $\alpha\geqslant 378$.

For $k = 9$ the 8-(24, 9, λ) is trivial. When $t = 7$ the matrix forces $\lambda = 8\alpha$. Also 7-(24, 9, 8α)$\Leftrightarrow$6-(24, 9, 48α). One obtains $\alpha = 8$ with (4, 0, 1), $\alpha = 60$ with (0, 4, 3), and eventually all $\alpha \geq 120$. With $t = 5$ then $\lambda = 12\alpha$. Also 5-(24, 9, 12α)$\Leftrightarrow$4-(24, 9, 48α)$\Leftrightarrow$3-(24, 9, 168α)$\Leftrightarrow$2-(24, 9, 528α). One gets $\alpha = 3$ with (1, 0, 0), $\alpha = 140$ with (0, 0, 1) and then all $\alpha \geq 278$.

With $k = 8$ then 7-(24, 8, λ)$\Leftrightarrow$6-(24, 8, 9λ). We get $\lambda = 15$ with (15, 0, 1) and by using the trivial design (1, 1, 1) for $\lambda = 17$ one eventually gets all nontrivial $\lambda \geq 239$. Now 5-(24, 8, α)$\Leftrightarrow$4-(24, 8, 5α)$\Leftrightarrow$3-(24, 8, 21α)$\Leftrightarrow$2-(24, 8, 77α). We get the classical Steiner system $S(5, 8, 24)$, with $\lambda = 1 = \alpha$, with (1, 0, 0) and hence all $\alpha \gtrsim 1$.

For $k = 7$ and $t = 6$ the designs here will be trivial. Now 5-(24, 7, 3α)$\Leftrightarrow$4-(24, 7, 20α)$\Leftrightarrow$3-(24, 7, 105α)$\Leftrightarrow$2-(24, 7, 462α) and these are attainable for all α using (1, 0) where $\alpha = 1$.

For $k = 6$, 5-(24, 6, α)$\Leftrightarrow$4-(24, 6, 10α)$\Leftrightarrow$3-(24, 6, 70α)$\Leftrightarrow$2-(24, 6, 385α). One gets $\alpha = 3$ with (1, 0), $\alpha = 16$ with (0, 1) and eventually all $\alpha \geq 39$.

For $k \geq 5$ we can get only trivial designs since M_{24} is quintuply transitive.

Designs with M_{23}

For $k = 11$ we have 10-(23, 11, α)$\Leftrightarrow$9-(23, 11, 7α)$\Leftrightarrow$8-(23, 11, 35α) and get $\alpha = 6$ with (6, 0, 0, 1, 0, 0, 6), $\alpha = 14$ with (14, 0, 0, 2, 1, 1, 2), and $\alpha = 17$ with (5, 1, 1, 2, 0, 0, 17). Using these solutions along with the trivial solution for $\alpha = 13$ we obtain all nontrivial solutions for $\alpha = 6$, 12, 14, 17, 18, 19, 20 and all $\alpha \geq 23$. When $t = 7$ the matrix forces $\lambda = 140\alpha$. We get $\alpha = 5$ with (5, 1, 0, 0, 0, 10, 50), $\alpha = 6$ with (6, 0, 0, 1, 0, 0, 6), $\alpha = 7$ with (7, 1, 0, 0, 1, 11, 40), and $\alpha = 9$ with (9, 1, 0, 0, 2, 12, 30). So we get $\alpha = 5, 6, 7$ and all $\alpha \geq 9$. With $t = 6$ the matrix forces $\lambda = 28\alpha$. One gets $\alpha = 5$ with (1, 0, 0, 0, 0, 0, 10) , $\alpha = 39$ with (3, 0, 0, 0, 1, 1, 0), $\alpha = 102$ with (6, 0, 0, 1, 0, 0, 6). We get all nontrivial designs for $\alpha \geq 137$. The matrix for 5-(23, 11, λ) forces $\lambda = 84\alpha$ and we get $\alpha = 4$ with (0, 0, 0, 0, 0, 1, 7), $\alpha = 5$ with (1, 0, 0, 0, 0, 0, 10) and so we obtain all nontrivial designs with $\lambda = 84\alpha$ for $\alpha = 4$, 5, 8, 9, 10 and all $\alpha \geq 12$. If $t = 4$ then $\lambda = 12\alpha$ and further 4-(23, 11, 12α)$\Leftrightarrow$3-(23, 11, 30α)$\Leftrightarrow$2-(23, 11, 70α). We get $\alpha = 4$ with (0, 0, 0, 0, 0, 0, 1), $\alpha = 55$ with (1, 0, 0, 0, 0, 0, 0) and eventually all $\alpha \geq 162$.

When $k = 10$ the designs here are all trivial for both $t = 8$ and 9. If $t = 7$, then the matrix forces $\lambda = 560\alpha$ and $\alpha = 1$ with (5, 1, 0, 6, 16) yields all $\alpha \geq 1$. With $t = 6$ the matrix implies $\lambda = 140\alpha$. We get $\alpha = 5$ with (1, 1, 0, 0, 10), $\alpha = 6$ with (2, 0, 0, 3, 3) and then all nontrivial designs for $\alpha = 5$, 6, 10, 11, 12, 15, 16, 17, 18, and all $\alpha \geq 20$. If $t = 5$, then the matrix implies $\lambda = 504\alpha$. We get $\alpha = 4$ with (0, 0, 0, 2, 7), $\alpha = 5$ with (1, 1, 0, 0, 10), $\alpha = 6$ with (2, 0, 0, 3, 3), and $\alpha = 7$ with (3, 1, 0, 1, 6). So we get all $\alpha \geq 4$. The matrix when $t = 4$ implies $\lambda = 84\alpha$. Here also 4-(23, 10, 84α)$\Leftrightarrow$3-(23, 10, 240α)$\Leftrightarrow$2-(23, 10, 630α). We get $\alpha = 4$ with (0, 0, 0, 0, 1), $\alpha = 15$ with (1, 0, 0, 0, 0) and then all $\alpha \geq 42$.

Let $k = 9$. The 8-(23, 9, λ) is here trivial. When $t = 7$ the matrix forces $\lambda = 120\alpha$. Using the trivial design and (32, 4, 0, 7, 13) where $\alpha = 4$ we get nontrivial designs for all $\alpha \geq 4$. When $t=6$ the matrix forces $\lambda = 40\alpha$. We get $\alpha = 6$ with $(2, 2, 0, 0, 3)$, $\alpha = 8$ with $(4, 0, 0, 1, 1)$, $\alpha = 21$ with $(1, 5, 1, 0, 6)$, and with the trivial design when $\alpha = 17$ we get all nontrivial designs with $\lambda = 40\alpha$ for $\alpha = 6$, 8, 12, 14, 16, 18 and all $\alpha \geq 20$. The matrix for $t=5$ forces $\lambda = 180\alpha$. One obtains $\alpha = 3$ with (11, 2, 0, 0, 0), $\alpha = 4$ with (8, 2, 0, 0, 1), $\alpha = 5$ with (5, 2, 0, 0, 2) and hence all $\alpha \geq 4$. For $t=4$ the matrix implies $\lambda = 36\alpha$. Further, 4-(23, 9, 36α)$\Leftrightarrow$3-(23, 9, 120α)$\Leftrightarrow$2-(23, 9, 360α). We get $\alpha = 3$ with (1, 0, 0, 0, 0), $\alpha = 28$ by (0, 0, 0, 0, 1), and then all $\alpha \geq 54$.

With $k=8$ and $t=7$ the matrix forces $\lambda = 16\alpha$. We get $\alpha = 4$ with $(64, 4, 0, 4, 5)$ so we get all $\alpha \geq 4$ using the trivial design. For $t=6$ the matrix requires $\lambda = 8\alpha$. We get $\alpha = 8$ with $(4, 4, 0, 0, 1)$, $\alpha = 15$ with $(15, 0, 0, 1, 1)$, and $\alpha = 34$ with $(2, 17, 1, 0, 4)$. Using the trivial design with $\alpha = 17$ we get all $\alpha \geq 53$. The matrix when $t=5$ forces $\lambda = 16\alpha$ and we get all α using $(11, 1, 0, 0, 0)$ where $\alpha = 1$. When $t=4$ the matrix forces $\lambda = 4\alpha$. Further, 4-(23, 8, 4α)$\Leftrightarrow$3-(23, 8, 20α)$\Leftrightarrow$2-(23, 8, 56α). One obtains all α using (1, 0, 0, 0, 0) where $\alpha = 1$.

Letting $k=7$ and $t=6$ we get $\lambda = 15$ with $(15, 0, 0, 1)$. The trivial design has $\lambda = 17$ so eventually all nontrivial designs are obtainable for $\lambda \geq 239$. For the t-(23, 7, λ) the minimal admissible $\lambda = 3$ is attained by $(3, 1, 0, 0)$ so all nontrivial $\lambda = 3\alpha$ are attained. Now 4-(23, 7, α)$\Leftrightarrow$3-(23, 7, 5α)$\Leftrightarrow$2-(23, 7, 21α) and we attain all α with $(1, 0, 0, 0)$ where $\alpha = 1$.

For $k=6$ the minimal λ for 5-(23, 6, λ) is $\lambda = 6$ and is attained using (3, 2, 0). Hence all nontrivial designs are obtained. Now 4-(23, 6, 3α)$\Leftrightarrow$3-(23, 6, 20α)$\Leftrightarrow$2-(23, 6, 105α) and we get all α using $(1, 0, 0)$ where $\alpha = 1$.

For $k=5$ here 4-(23, 5, α)$\Leftrightarrow$3-(23, 5, 10α)$\Leftrightarrow$2-(23, 5, 70α). We get $\alpha = 3$ with (1, 0), $\alpha = 16$ with (0, 1) and then all $\alpha \geq 39$.

If $k \leq 4$ all designs are trivial because M_{23} is quadruply transitive.

Designs with M_{22}

For the 10-(22, 11, λ) we need $\lambda = 12\alpha$ and we get $\alpha = 2$ with (0, 4, 2, 2, 2, 3, 0, 1, 2, 4, 4, 0, 2, 2, 0, 4) and hence produce all nontrivial designs for $\alpha \geq 2$. When $t=9$ the minimal λ is 6 so $\lambda = 6\alpha$. Also 9-(22, 11, 6α)$\Leftrightarrow$8-(22, 11, 28α). We get $\alpha = 6$ with (6, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 6, 0), $\alpha = 11$ with (7, 1, 1, 0, 0, 0, 2, 1, 2, 0, 2, 0, 1, 1, 0, 0), $\alpha = 14$ using (8, 6, 0, 0, 0, 1, 0, 1, 2, 2, 1, 1, 1, 1, 1, 2), $\alpha = 15$ with (3, 9, 1, 0, 0, 1, 3, 1, 3, 0, 0, 0, 0, 0, 0, 15), and $\alpha = 16$ with (2, 0, 2, 1, 1, 1, 2, 1, 2, 0, 3, 0, 1, 1, 5, 0). These together with the trivial design (when $\alpha = 13$) produce nontrivial designs for $\lambda = 6\alpha$ when $\alpha = 6, 11, 12$ and all $\alpha \geq 14$. For $t=7$ then $\lambda = 105\alpha$ and we get $\alpha = 3$ with (3, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 3, 3, 15, 10), $\alpha = 7$ with (1, 1, 3, 0, 0, 1, 0, 1, 0, 0, 0, 0, 2, 2, 20, 20), and $\alpha = 8$ with (8, 2, 2, 0, 0, 0, 0, 0, 0, 0, 3, 1, 7, 7, 31, 10). So we get $\alpha = 3$ and all $\alpha \geq 6$. When $t=6$ the matrix forces $\lambda = 336\alpha$. Now a 7-(22, 11, 105α) is a 6-(22, 11, 336α) so we have solutions for $\alpha = 3$ and 7. We get $\alpha = 5$ with (5, 1, 1, 0, 0, 0, 0, 0, 0, 0, 2, 1, 4, 4, 16, 0) so we get $\alpha = 3$ and all $\alpha \geq 5$. For $t=5$ the matrix forces $\lambda = 56\alpha$. We get $\alpha = 5$ with (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 10), $\alpha = 8$ with (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 8, 8) and eventually all $\alpha \geq 28$. For $t = 4$ the matrix forces $\lambda = 144\alpha$. We get $\alpha = 4$ with (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 8), $\alpha = 5$ with (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 10). So we get nontrivial designs with $\lambda = 144\alpha$ for $\alpha = 4, 5, 8, 9, 10$ and all $\alpha \geq 12$. If $t = 3$ then the matrix implies $\lambda = 18\alpha$. Also 3-(22, 11, 18α)$\Leftrightarrow$2-(22, 11, 40α). We get $\alpha = 4$ using the last orbit and $\alpha = 55$ with the first orbit. Eventually all $\alpha \geq 162$ are obtained.

Now 9-(22, 10, λ)$\Leftrightarrow$8-(22, 10, 7λ) and we get $\lambda = 6$ with (6, 6, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 6), $\lambda = 14$ with (14, 14, 0, 0, 0, 0, 0, 2, 2, 1, 1, 1, 2), $\lambda = 17$ with (5, 5, 1, 1, 1, 1, 1, 2, 2, 0, 0, 0, 17). With the trivial design when $\lambda = 13$ these produce all nontrivial designs for $\lambda = 6$, 12, 14, 17, 18, 19, 20 and all $\lambda \geq 23$. For $t = 7$ we have $\lambda = 35\alpha$ and we get $\alpha = 3$ with (3, 3, 1, 0, 0, 0, 0, 0, 2, 0 3, 3, 12), $\alpha = 5$ with (5, 5, 1, 0, 0, 0, 0, 0, 2, 1, 4, 4, 2). These yield nontrivial designs for $\alpha = 3$, 5, 6 and all $\alpha \geq 8$. For $t = 6$ the matrix forces $\lambda = 140\alpha$. Now a 7-(22, 10, 35α) is a 6-(22, 10, 140α) so we have solutions for $\alpha = 3$ and 5. We also get $\alpha = 7$ with (11, 1, 1, 1, 1, 0, 1, 0, 0, 0, 6, 6, 55) so we get $\alpha = 3$ and all $\alpha \geq 5$. When $t = 5$ the matrix requires $\lambda = 28\alpha$. We get $\alpha = 5$ with (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 10), $\alpha = 12$ with (4, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 6), and $\alpha = 28$ with (0, 2, 0, 0, 0, 0, 1, 0, 0, 0, 2, 2, 35). we eventually get all $\alpha \geq 32$. For $t = 4$ the matrix forces $\lambda = 84\alpha$. We get $\alpha = 4$ with (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 7), $\alpha = 5$ with (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 10), $\alpha = 6$ with (2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 21), and hence all $\alpha \geq 8$. If $t = 3$ the matrix forces $\lambda = 12\alpha$. Further 3-(22, 10, 12α)$\Leftrightarrow$2-(22, 10, 30α). We get $\alpha = 4$ with the last orbit, $\alpha = 15$ with the first orbit and hence all $\alpha \geq 42$.

When $k = 9$ the 8-(22, 9, λ) here will be trivial. For $t = 7$ then $\lambda = 105\alpha$ and we get $\alpha = 2$ with (14, 2, 2, 2, 2, 0, 3, 5, 5, 11) and hence all $\alpha \geq 2$. For $t = 6$ the matrix implies $\lambda = 560\alpha$. We get $\alpha = 1$ with (5, 1, 5, 1, 1, 0, 0, 6, 0, 16) and hence all α. For $t = 5$ the matrix forces $\lambda = 140\alpha$. We get $\alpha = 3$ with (3, 1, 1, 1, 0, 0, 0, 0, 0, 3) and $\alpha = 5$ with (1, 1, 1, 1, 1, 0, 0, 0, 0, 10). They produce nontrivial designs for $\alpha = 3, 5, 6$ and all $\alpha \geq 8$. For $t = 4$ the matrix implies $\lambda = 504\alpha$. We get $\alpha = 2$ with (6, 0, 0, 0, 0, 0, 0, 0, 0, 5), $\alpha = 3$ with (3, 0, 0, 0, 0, 0, 0, 0, 1, 6) and hence all $\alpha \geq 2$. If $t = 3$ then the matrix forces $\lambda = 84\alpha$. Also 3-(22, 9, 84α)$\Leftrightarrow$2-(22, 9, 240α). We get $\alpha = 3$ with the first orbit, $\alpha = 4$ with the last orbit and then all $\alpha \geq 6$.

If $k = 8$ and $t = 7$ then $\lambda = 15\alpha$. We get all nontrivial solutions for $\alpha \geq 2$ using the trivial solution when $\alpha = 1$ and the solution (30, 2, 2, 2, 0, 2, 2, 2, 3, 3) where $\alpha = 2$. If $t = 6$ the matrix implies $\lambda = 120\alpha$. We get $\alpha = 2$ with (6, 0, 0, 2, 2, 4, 1, 1, 4, 1) and hence all $\alpha \geq 2$ using the trivial solution for $\alpha = 1$. When $t = 5$ the matrix forces $\lambda = 40\alpha$. We get $\alpha = 4$ with (36, 5, 5, 0, 0, 0, 0, 0, 1, 0), $\alpha = 6$ with (110, 11, 11, 2, 0, 0, 0, 0, 0, 0), $\alpha = 7$ with (7, 0, 0, 1, 0, 1, 0, 0, 1, 1), and $\alpha = 9$ with (9, 0, 0, 3, 0, 1, 0, 0, 0, 3). So we get $\alpha = 4$ and all $\alpha \geq 6$. For $t = 4$ the matrix yields $\lambda = 60\alpha$ and we get all α using $\alpha = 1$ with (11, 0, 1, 0, 0, 0, 0, 0, 0, 0). When $t = 3$

the matrix forces $\lambda = 12\alpha$. Also 3-$(22, 8, 12\alpha) \Leftrightarrow$ 2-$(22, 8, 40\alpha)$. We get $\alpha = 1$ using the first orbit and hence all α.

When $k = 7$ the matrix for $t = 6$ forces $\lambda = 16\alpha$. We get $\alpha = 4$ with $(64, 4, 4, 0, 0, 4, 4, 5)$ and with the trivial design when $\alpha = 1$ we get all $\alpha \geqslant 4$. When $t = 5$ the matrix requires $\lambda = 8\alpha$. We get $\alpha = 2$ with $(16, 16, 1, 5, 0, 0, 0, 0)$ and $\alpha = 15$ with $(90, 90, 0, 24, 0, 0, 1, 0)$. So we get $\alpha = 2, 4, 6, 8, 10, 12$ and all $\alpha \geqslant 14$. The matrix when $t = 4$ implies $\lambda = 16\alpha$ and we get all α using $(2, 2, 0, 1, 0, 0, 0, 0)$ for $\alpha = 1$. For $t = 3$ the matrix requires $\lambda = 4\alpha$. Further 3-$(22, 7, 4\alpha) \Leftrightarrow$ 2-$(22, 7, 16\alpha)$. We get all α using the first orbit when $\alpha = 1$.

When $k = 6$ and $t = 5$ we get $\lambda = 6$ with $(6, 3, 3, 0, 2, 0)$, $\lambda = 15$ with $(15, 0, 0, 0, 1, 1)$, $\lambda = 20$ with $(4, 10, 10, 1, 6, 0)$, and $\lambda = 34$ with $(2, 17, 17, 2, 10, 0)$. Hence we produce all $\lambda \geq 32$. For $t = 4$ the minimal $\lambda = 3$ is attained using $(3, 0, 1, 0, 0, 0)$ and so all possible λ are attained. Now here 3-$(22, 6, \lambda) \Leftrightarrow$ 2-$(22, 6, 5\lambda)$ and $\lambda = 1$ is obtained using the first orbit and hence all λ are obtained.

When $k = 5$ and $t = 4$ the minimal $\lambda = 6$ is obtained with $(3, 1, 1, 0)$ so all possible λ occur. If $t = 3$ the minimal $\lambda = 3$ is attained with the first orbit. Also 3-$(22, 5, 3\alpha) \Leftrightarrow$ 2-$(22, 5, 20\alpha)$ so all α occur.

When $k = 4$ then 3-$(22, 4, \alpha) \Leftrightarrow$ 2-$(22, 4, 10\alpha)$. We get $\alpha = 3$ with $(1, 0)$, $\alpha = 16$ with $(0, 1)$ and then all $\alpha \geqslant 39$.

If $k \leqslant 3$ all the designs are trivial since M_{22} is triply transitive.

5. Concluding remarks

The basic purpose of this paper was to determine the action of the large Mathieu groups on an appropriate power set, and then to use that information to search mainly for t-designs (allowing repeated blocks) with $t \geqslant 6$. Finding such designs for even moderately small λ was fairly easy with our methods. It was a mild disappointment not to have found such designs for $6 \leqslant t$ with no repeated blocks.

It was expedient to limit the scope of this paper to the action and designs of M_{24}, M_{23}, and M_{22}, but our methods will certainly allow us to determine the action of $M_{21} \cong \mathrm{PSL}_3(4)$ on $\mathcal{P}(\Omega_{21})$, $\Omega_{21} = \Omega_{22} \setminus \{1\}$. The associated diagram will be somewhat unwieldly, however, requiring as many as 45 orbits of 10-sets. This work is in progress but a few of the associated matrices will be somewhat more difficult to completely search for t-designs.

References

[1] W.O. Alltop, An infinite class of 5-designs, J. Combinatorial Theory (A) 12 (1972) 390–395.
[2] A.E. Brouwer, The t-designs with $v < 18$, Mathematical Centre, 2e Boerhaavestraat 49, Amsterdam, Netherlands (1977).

[3] Chang Choi, The subgroup structure of the Mathieu group of degree 24, Ph.D. Thesis, Univ. of Michigan, Ann Arbor, MI, 1968.
[4] Chang Choi, On subgroups of M_{24}, I: Stabilizers of subsets, Trans. Amer. Math. Soc. 167 (1972) 1–27.
[5] R.H.F. Denniston, Some new 5-designs, Bull. London Math. Soc. 8 (1976), 263–267.
[6] John H. Conway, Three lectures on exceptional groups, in: M.B. Powell and G. Higman, eds., Finite Simple Groups (Academic Press, New York, 1971) 215–247.
[7] X. Hubaut, Two new families of 4-designs, Discrete Math. 9 (1974) 247–249.
[8] E.S. Kramer, Some t-designs for $t \geqslant 4$ and $v = 17$, 18, Proc. 6th South-Eastern Conf. Combinatorics, Graph Theory and Computing (Boca Raton 1975), Congressus Numerantium XIV (Utilitas Math. Publ., Winnipeg, 1975) 443–460.
[9] E.S. Kramer and D.M. Mesner, t-designs on hypergraphs, Discrete Math. 15 (1976) 263–296.
[10] D. Livingstone and A. Wagner, Transitivity of finite permutation groups on unordered sets, Math. Z. 90 (1965) 393–403.
[11] G.M. Saha, Some results in tactical configurations and related topics, Utilitas Math. 7 (1975) 223–240.
[12] J.A. Todd, A representation of the Mathieu group M_{24} as a collineation group, Ann. Mat. Pura Appl. (4) 71 (1966) 199–238.
[13] R.M. Wilson, The necessary conditions for t-designs are sufficient for something, Utilitas Math. 4 (1973) 207–215.